

COMPARISON OF ESTIMATES FOR DISPERSIVE EQUATIONS

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ABSTRACT. This paper describes a new comparison principle that can be used for the comparison of space-time estimates for dispersive equations. In particular, results are applied to the global smoothing estimates for several classes of dispersive partial differential equations.

1. INTRODUCTION

In this note we will present a new comparison principle that allows one to compare certain estimates for dispersive equations of different types based on expressions involving their symbols. In particular, a question is that if we have a certain estimate for one equation, whether we can derive a corresponding estimate for another equation. This question is of interest on its own, and it has several applications. Proofs of the statements of this paper can be found in the authors' paper [14].

The main application of this technique that we have in mind is for the global smoothing estimates for dispersive equations. These smoothing estimates are essentially global space-time estimates in weighted Sobolev spaces over L^2 , see for example [2, 3, 5, 7, 16, 17]. There is a known method in the microlocal analysis on how to transform one equation into another, namely the canonical transforms realised in the form of Fourier integral operators [6]. In the global setting one needs to develop global weighted estimates in L^2 for the corresponding classes of Fourier integral operators in order to apply them to the smoothing estimates. Such global estimates for Fourier integral operators have been established by the authors [12] and have been applied to derive new smoothing estimates for Schrödinger equations [12, 13]. These techniques allow one to reduce the analysis of dispersive equations to normal forms in one and two dimensions [11]. The comparison principles introduced below allow one to further relate estimates in normal forms, thus establishing comprehensive relations between smoothing estimates for dispersive equations with constant coefficient [14].

In this note, we denote $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$, and $D_x = (D_1, D_2, \dots, D_n)$, where D_j denotes $D_{x_j} = \frac{1}{i} \frac{\partial}{\partial x_j}$, for all $j = 1, 2, \dots, n$, and $i = \sqrt{-1}$.

2. COMPARISON PRINCIPLES

Let $u(t, x) = e^{itf(D_x)}\varphi(x)$ and $v(t, x) = e^{itg(D_x)}\varphi(x)$ be solutions to the following evolution equations, where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$:

$$(2.1) \quad \begin{cases} (i\partial_t + f(D_x)) u(t, x) = 0, \\ u(0, x) = \varphi(x), \end{cases}$$

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and

$$(2.2) \quad \begin{cases} (i\partial_t + g(D_x)) v(t, x) = 0, \\ v(0, x) = \varphi(x). \end{cases}$$

First we state the following result relating several norms involving propagators for equations (2.1) and (2.2):

Theorem 2.1. *Let $f \in C^1(\mathbb{R}^n)$ be a real-valued function such that, for almost all $\xi' = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1}$, $f(\xi_1, \xi')$ is strictly monotone in ξ_1 on the support of a measurable function σ on \mathbb{R}^n . Then we have*

$$\|\sigma(D_x)e^{itf(D_x)}\varphi(x_1, x')\|_{L^2(\mathbb{R}_t \times \mathbb{R}_{x'}^{n-1})}^2 = (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{\varphi}(\xi)|^2 \frac{|\sigma(\xi)|^2}{|\partial f / \partial \xi_1(\xi)|} d\xi$$

for all $x_1 \in \mathbb{R}$, where $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$.

The following comparison principle is a straightforward consequence of Theorem 2.1:

Corollary 2.2. *Let $f, g \in C^1(\mathbb{R}^n)$ be real-valued functions such that, for almost all $\xi' = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1}$, $f(\xi_1, \xi')$ and $g(\xi_1, \xi')$ are strictly monotone in ξ_1 on the support of a measurable function χ on \mathbb{R}^n . Let $\sigma, \tau \in C^0(\mathbb{R}^n)$ be such that, for some $A > 0$, we have*

$$(2.3) \quad \frac{|\sigma(\xi)|}{|\partial_{\xi_1} f(\xi)|^{1/2}} \leq A \frac{|\tau(\xi)|}{|\partial_{\xi_1} g(\xi)|^{1/2}}$$

for all $\xi \in \text{supp } \chi$ satisfying $D_1 f(\xi) \neq 0$ and $D_1 g(\xi) \neq 0$. Then we have

$$(2.4) \quad \begin{aligned} \|\chi(D_x)\sigma(D_x)e^{itf(D_x)}\varphi(x_1, x')\|_{L^2(\mathbb{R}_t \times \mathbb{R}_{x'}^{n-1})} \\ \leq A \|\chi(D_x)\tau(D_x)e^{itg(D_x)}\varphi(\tilde{x}_1, x')\|_{L^2(\mathbb{R}_t \times \mathbb{R}_{x'}^{n-1})} \end{aligned}$$

for all $x_1, \tilde{x}_1 \in \mathbb{R}$, where $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$. Consequently, for any measurable function w on \mathbb{R} we have

$$(2.5) \quad \begin{aligned} \|w(x_1)\chi(D_x)\sigma(D_x)e^{itf(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \\ \leq A \|w(x_1)\chi(D_x)\tau(D_x)e^{itg(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)}. \end{aligned}$$

Moreover, if $\chi \in C^0(\mathbb{R}^n)$ and $w \neq 0$ on a set of \mathbb{R} with positive measure, the converse is true, namely, if we have estimate (2.4) for all φ , for some $x_1, \tilde{x}_1 \in \mathbb{R}$, or if we have estimate (2.5) for all φ , and the norms are finite, then we also have inequality (2.3).

We remark that inequality (2.5) in Corollary 2.2 gives the comparison between different weighted estimates. The reason to introduce a cut-off function χ into the estimates is that the relation between symbols may be different for different regions of the frequencies ξ (for example this is the case for the relativistic Schrödinger and for the Klein-Gordon equations, which are of order two for large frequencies and of order zero for small frequencies). In such case we can use this comparison principle to relate estimates for the corresponding ranges of frequencies, thus yielding more

refined results, since then we have freedom to choose different σ for different types of behaviour of f' . The assumption $\sigma, \tau \in C^0(\mathbb{R}^n)$ that was made in Corollary 2.2 is for the clarity of the exposition and can clearly be relaxed.

In the case $n = 1$, we neglect $x' = (x_2, \dots, x_n)$ in a natural way and just write $x = x_1$, $\xi = \xi_1$, and $D_x = D_1$. Similarly in the case $n = 2$, we use the notation $(x, y) = (x_1, x_2)$, $(\xi, \eta) = (\xi_1, \xi_2)$, and $(D_x, D_y) = (D_1, D_2)$. In both cases, we write $\tilde{x} = \tilde{x}_1$ in the notation of Corollary 2.2. Then we have the following corollaries in lower dimensions:

Corollary 2.3. *Suppose $n = 1$. Let $f, g \in C^1(\mathbb{R})$ be real-valued and strictly monotone on the support of a measurable function χ on \mathbb{R} . Let $\sigma, \tau \in C^0(\mathbb{R})$ be such that, for some $A > 0$, we have*

$$(2.6) \quad \frac{|\sigma(\xi)|}{|f'(\xi)|^{1/2}} \leq A \frac{|\tau(\xi)|}{|g'(\xi)|^{1/2}}$$

for all $\xi \in \text{supp } \chi$ satisfying $f'(\xi) \neq 0$ and $g'(\xi) \neq 0$. Then we have

$$(2.7) \quad \|\chi(D_x)\sigma(D_x)e^{itf(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t)} \leq A\|\chi(D_x)\tau(D_x)e^{itg(D_x)}\varphi(\tilde{x})\|_{L^2(\mathbb{R}_t)}$$

for all $x, \tilde{x} \in \mathbb{R}$. Consequently, for general $n \geq 1$ and for any measurable function w on \mathbb{R}^n , we have

$$(2.8) \quad \|w(x)\chi(D_j)\sigma(D_j)e^{itf(D_j)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \\ \leq A\|w(x)\chi(D_j)\tau(D_j)e^{itg(D_j)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)},$$

where $j = 1, 2, \dots, n$. Moreover, if $\chi \in C^0(\mathbb{R})$ and $w \neq 0$ on a set of \mathbb{R}^n with positive measure, the converse is true, namely, if we have estimate (2.7) for all φ , for some $x, \tilde{x} \in \mathbb{R}$, or if we have estimate (2.7) for all φ , and the norms are finite, then we also have inequality (2.6).

We have the following comparison principle in two dimensions:

Corollary 2.4. *Suppose $n = 2$. Let $f, g \in C^1(\mathbb{R}^2)$ be real-valued functions such that, for almost all $\eta \in \mathbb{R}$, $f(\xi, \eta)$ and $g(\xi, \eta)$ are strictly monotone in ξ on the support of a measurable function χ on \mathbb{R}^2 . Let $\sigma, \tau \in C^0(\mathbb{R}^2)$ be such that, for some $A > 0$, we have*

$$(2.9) \quad \frac{|\sigma(\xi, \eta)|}{|\partial f / \partial \xi(\xi, \eta)|^{1/2}} \leq A \frac{|\tau(\xi, \eta)|}{|\partial g / \partial \xi(\xi, \eta)|^{1/2}}$$

for all $(\xi, \eta) \in \text{supp } \chi$ satisfying $\partial f / \partial \xi(\xi, \eta) \neq 0$ and $\partial g / \partial \xi(\xi, \eta) \neq 0$. Then we have

$$(2.10) \quad \|\chi(D_x, D_y)\sigma(D_x, D_y)e^{itf(D_x, D_y)}\varphi(x, y)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \\ \leq A\|\chi(D_x, D_y)\tau(D_x, D_y)e^{itg(D_x, D_y)}\varphi(\tilde{x}, y)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)}$$

for all $x, \tilde{x} \in \mathbb{R}$. Consequently, for general $n \geq 2$ and for any measurable function w on \mathbb{R}^{n-1} we have

$$(2.11) \quad \|w(\tilde{x}_k)\chi(D_j, D_k)\sigma(D_j, D_k)e^{itf(D_j, D_k)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \\ \leq A\|w(\tilde{x}_k)\chi(D_j, D_k)\tau(D_j, D_k)e^{itg(D_j, D_k)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)},$$

where $j \neq k$ and $\tilde{x}_k = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$. Moreover, if $\chi \in C^0(\mathbb{R}^2)$ and $w \neq 0$ on a set of \mathbb{R}^{n-1} with positive measure, the converse is true, namely, if we have estimate (2.10) for all φ , for some $x, \tilde{x} \in \mathbb{R}$, or if we have estimate (2.10) for all φ , and the norms are finite, then we also have inequality (2.9).

By the same argument as used in the proof of Theorem 2.1 and Corollary 2.2, we have a comparison result in the radially symmetric case. Below, we denote the set of the positive real numbers $(0, \infty)$ by \mathbb{R}_+ .

Theorem 2.5. *Let $f, g \in C^1(\mathbb{R}_+)$ be real-valued and strictly monotone on the support of a measurable function χ on \mathbb{R}_+ . Let $\sigma, \tau \in C^0(\mathbb{R}_+)$ be such that, for some $A > 0$, we have*

$$(2.12) \quad \frac{|\sigma(\rho)|}{|f'(\rho)|^{1/2}} \leq A \frac{|\tau(\rho)|}{|g'(\rho)|^{1/2}}$$

for all $\rho \in \text{supp } \chi$ satisfying $f'(\rho) \neq 0$ and $g'(\rho) \neq 0$. Then we have

$$(2.13) \quad \|\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t)} \leq A \|\chi(|D_x|)\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t)}$$

for all $x \in \mathbb{R}^n$. Consequently, for any measurable function w on \mathbb{R}^n , we have

$$(2.14) \quad \|w(x)\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq A \|w(x)\chi(|D_x|)\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)}.$$

Moreover, if $\chi \in C^0(\mathbb{R}_+)$ and $w \neq 0$ on a set of \mathbb{R}^n with positive measure, the converse is true, namely, if we have estimate (2.13) for all φ , for some $x \in \mathbb{R}^n$, or if we have estimate (2.14) for all φ , and the norms are finite, then we also have inequality (2.12).

Theorem 2.5 provides an analytic alternative to computations for certain estimates in the radially symmetric case done with the help of special functions [18].

These comparison principles can be extended to provide the relation between Strichartz type norms, and the details and the meaning of the corresponding estimates can be found in authors' paper [14]. Here we just give one corollary:

Corollary 2.6. *Let functions f, g, σ, τ be as in Theorem 2.5 and satisfy relation (2.12). Let $0 < p \leq \infty$. Then, for any measurable function w on \mathbb{R}^n , we have the estimate*

$$(2.15) \quad \|w(x)\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^p(\mathbb{R}_x^n, L^2(\mathbb{R}_t))} \leq A \|w(x)\chi(|D_x|)\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^p(\mathbb{R}_x^n, L^2(\mathbb{R}_t))}.$$

From this, it follows, for example, that for all $0 < p \leq \infty$, quantities $\|e^{it\sqrt{-\Delta}}\varphi\|_{L^p(\mathbb{R}_x^n, L^2(\mathbb{R}_t))}$, $\||D_x|^{1/2}e^{-it\Delta}\varphi\|_{L^p(\mathbb{R}_x^n, L^2(\mathbb{R}_t))}$, and $\||D_x|e^{it(-\Delta)^{3/2}}\varphi\|_{L^p(\mathbb{R}_x^n, L^2(\mathbb{R}_t))}$ for the propagators of the wave, Schrödinger, and KdV type equations are equivalent.

3. SOME APPLICATIONS

Let us now give some examples of the use of these comparison principles. If both sides in expression (2.3) in Corollary 2.2 are equivalent, we can use the comparison

in two directions, from which it follows that norms on both sides in (2.4) are equivalent. The same is true for Corollaries 2.3, 2.4 and Theorem 2.5. In particular, we can conclude that many smoothing estimates for the Schrödinger type equations of different orders are equivalent to each other. Indeed, applying Corollary 2.3 in two directions, we immediately obtain that for $n = 1$ and $l, m > 0$, we have

$$(3.1) \quad \left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} = \sqrt{\frac{l}{m}} \left\| |D_x|^{(l-1)/2} e^{it|D_x|^l} \varphi(x) \right\|_{L^2(\mathbb{R}_t)}$$

for every $x \in \mathbb{R}$, assuming that $\text{supp } \widehat{\varphi} \subset [0, +\infty)$ or $(-\infty, 0]$. On the other hand, still in the case $n = 1$, we have easily

$$(3.2) \quad \left\| e^{itD_x} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} = \|\varphi\|_{L^2(\mathbb{R}_x)} \quad \text{for all } x \in \mathbb{R},$$

which is a straightforward consequence of the fact $e^{itD_x} \varphi(x) = \varphi(x + t)$. These observations yield:

Theorem 3.1. *Suppose $n = 1$ and $m > 0$. Then we have*

$$(3.3) \quad \left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x)}$$

for all $x \in \mathbb{R}$. Suppose $n = 2$ and $m > 0$. Then we have

$$(3.4) \quad \left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \leq C \|\varphi\|_{L^2(\mathbb{R}_{x,y}^2)}$$

for all $x \in \mathbb{R}$. Each estimate above is equivalent to itself with $m = 1$ which is a direct consequence of equality (3.2).

Estimates (3.3) and (3.4) in Theorem 3.1 in the special case $m = 2$ were shown by Kenig, Ponce and Vega [9, p.56] and by Linares and Ponce [10, p.528], respectively. Theorem 3.1 shows that these results, together with their generalisation to other orders m , are in fact just corollaries of the elementary one dimensional fact $e^{itD_x} \varphi(x) = \varphi(x + t)$ once we apply the comparison principle.

By using the comparison principle in the radially symmetric and higher dimensional cases, we have also another type of equivalence of smoothing estimates, which can be found in authors' paper [14]. Let us give one example:

Theorem 3.2. *For $m > 0$ (and any α, β) we have the following relations (in the first equality the left and the right hand sides are finite for the same values of α, β at the same time)*

$$\begin{aligned} & \left\| |x|^{\beta-1} |D_x|^\beta e^{it|D_x|^2} \varphi \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} = \\ & \quad \sqrt{\frac{m}{2}} \left\| |x|^{\beta-1} |D_x|^{m/2+\beta-1} e^{it|D_x|^m} \varphi \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)}, \\ & \quad \left\| \langle x \rangle^{\alpha-m/2} |D_x|^\alpha e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \\ & \leq \left\| |x|^{\alpha-m/2} |D_x|^\alpha e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \\ & \leq \sup_{\lambda > 0} \left\| \langle x \rangle^{\alpha-m/2} |D_x|^\alpha e^{it|D_x|^m} \varphi_\lambda(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)}, \end{aligned}$$

where $\varphi_\lambda(x) = \lambda^{n/2} \varphi(\lambda x)$, and we take $\alpha \leq m/2$ in the last estimate. The operator norms of operators $\langle x \rangle^{\alpha-m/2} |D_x|^\alpha e^{it|D_x|^m}$ and $|x|^{\alpha-m/2} |D_x|^\alpha e^{it|D_x|^m}$ as mappings from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}_t \times \mathbb{R}_x^n)$ are equal.

As a nice consequence, for $n \geq 3$ and $m > 0$ we can conclude also the estimate

$$(3.5) \quad \left\| |x|^{-1} |D_x|^{m/2-1} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq \sqrt{\frac{2\pi}{m(n-2)}} \|\varphi\|_{L^2(\mathbb{R}_x^n)},$$

where the constant $\sqrt{\frac{2\pi}{m(n-2)}}$ is sharp. This follows from the first equality in Theorem 3.2 with $\beta = 0$ and the best constant in the case $m = 2$ (as shown by Simon [15] as a consequence of constants in Kato's theory).

As a consequence of Theorem 3.2, we have

Corollary 3.3. *Suppose $m > 0$ and $(m-n)/2 < \alpha < (m-1)/2$. Then we have*

$$(3.6) \quad \left\| |x|^{\alpha-m/2} |D_x|^\alpha e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

Suppose $m > 0$ and $(m-n+1)/2 < \alpha < (m-1)/2$. Then we have

$$(3.7) \quad \left\| |x|^{\alpha-m/2} |D'|^\alpha e^{it(|D_1|^m - |D'|^m)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)},$$

where $D' = (D_2, \dots, D_n)$.

Estimate (3.6) is known in the case $m = 2$ as the Kato–Yajima estimate [8]. The application of the comparison principles also yields some refinements for other equations, for example for the relativistic Schrödinger equation [4], Klein-Gordon equations and wave equations [1]. We refer to [14] for further details.

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